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Experimental approach to the study of complex network synchronization using a single oscillator

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We propose an experimental setup based on a single oscillator for studying large networks formed by identical unidirectionally coupled systems. A chaotic wave form generated by the oscillator is stored in a computer to adjust the signal according to the desired network configuration to feed it again into the same oscillator. No previous theoretical knowledge about the oscillator dynamics is needed. To visualize network synchronization we introduce a network synchronization bifurcation diagram that should prove to be an effective tool for analysis, design, and optimization of complex networks.

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A very interesting problem today is synchronization of a large number of dynamical systems connected together in complex networks [1–4]. From molecular biology to neuroscience, from condensed-matter physics to the internet, researchers are unraveling the structure of complex networks, learning how they evolve and function, and exploring how their architecture affects the collective behavior they display. One simple way to coordinate the motions of many systems is to completely synchronize them. It is important to understand how the synchronizability of an oscillators' network depends on the interconnection of those oscillators. Much has been learned by studying this problem theoretically either from a local perspective, using linearization to examine how the network topology affects the stability spectrum for the synchronous state [5–12] or by focusing on a more global property of phase space: the basin of attraction for the synchronous state [13], sadly an exact analytical determination of synchronization condition can be done only in some simple cases of coupling configurations (symmetrical or global coupling) [14]. The more complex the network structure, the more sophisticated the numerical techniques necessary become, making an experimental approach very alluring.

The experimental study of complex networks is an extremely difficult task. For example, in neuroscience it is usually restricted to collecting and analyzing experimental data recorded from different neuron groups [15] and in sociology it is usually limited by searching the relationship between individuals [16]. Presently, one difficult problem is to set up a real physical experiment to study synchronization dynamics of large networks with predesigned configuration. Many open questions, important in many real-world settings, such as under what conditions will such a network fall into sync with all its elements acting as one, how does a network's ability to self-synchronize depend on its wiring diagram, and how to choose the optimal topology for achieving the desired synchronization state, until now have been approachable only by numerical simulations. The influence of network topology on the stability of a synchronized chaotic motion is currently a hot research topic. For example, collective dynamics of coupled electronic circuits replicating artificial neurons, has recently been used to control the movement of a biomimetic robot [17]. Furthermore, all physical and mental functionings depend on the establishment and maintenance of large neuron networks: the human brain contains as many as 100 billion neurons interconnected in complex networks [18]. The lack of information we possess on such networks makes theoretical research very hard, leaving us no other choice than to undertake the experimental path. Thus, the practical realization of a complex network with a huge number of chaotic oscillators might be a step toward the creation of artificial intelligence and toward better understanding of the brain functions. The traditional method for constructing complex networks poses serious technical problems that have restricted the experimental study to relatively small networks.

In this Rapid Communication we propose an experimental arrangement based on a single oscillator to study large complex networks of unidirectionally interacting units with a desired topology. A chaotic wave form generated by the oscillator passes through an analog-to-digital converter (ADC) and is stored in a computer. Then, after passing through a digital-to-analog converter (DAC) and an operational amplifier, the digital signal from the computer, is inserted back into the same oscillator to provide the coupling between nodes, which form a network with a predefined configuration. The sketch of the experimental setup is shown in Fig. 1. It consists of a nonlinear oscillator, an ADC, a personal computer (PC), a DAC, and an operational amplifier (OA). First, the oscillator acts as node 1, then the oscillator's analog out-

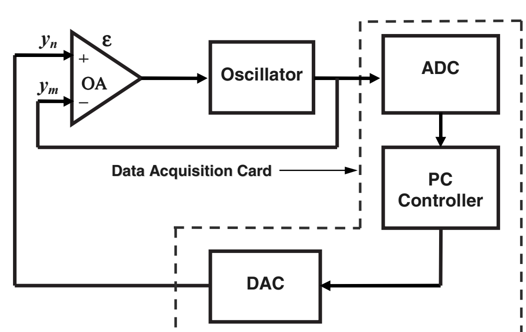

FIG. 1. Experimental setup for studying complex network synchronization. ADC and DAC are analogical-to-digital and digital-to-analogical converters, PC is a personal computer, and OA is an operational amplifier.

put passes through ADC and is stored in PC, where it is analyzed with a special program and after attenuation, goes through DAC, and, according to the coupling strength between node 1 and node 2, enters back into the oscillator. The oscillator, then, acts as node 2 and its output is stored again in PC. This process is repeated successively depending on the number of nodes in the designed network. If a node has several links, the computer adjusts the input signal according to the connectivity matrix. This experimental setup simulates a network of identical or nearly identical dynamical systems. Since the wave forms are stored in the computer and then injected back into the analog system as an external driving signal and the oscillator is self-oscillating, the node needs time to be synchronized. To be able to neglect this time, the stored time series should be much larger than transient time to achieve pairwise synchronization. In our experiments, the duration of the recorded wave forms is about 3 orders of magnitude larger than the synchronization time.

The approach of recording the output of a single system in a memory to study synchronization between two unidirectionally coupled identical chaotic units was introduced by Pyragas [19] and experimentally demonstrated with a chaotic electronic circuit [20]. In the present work we extend this idea to a network of coupled oscillators. The general equation of motion of a network build by N unidirectionally linearly coupled identical oscillators can be written as

$$\dot{x}_n = F(x_n) + \varepsilon \sum_{m=2}^{N} c_{nm} P x_m, \quad n = 1, \ldots, N, \quad n \neq m. \quad (1)$$

Here, $x_n=(x_n^1,\ldots,x_n^d)$ is the d-dimensional vector in the phase space of the nth oscillator, $F(x_n)$ is a nonlinear vector function that determines the oscillator dynamics, $\varepsilon \in [0,1]$ is the coupling strength (for simplicity, we suppose that ε is the same for every link), c_{nm} is the element of the connectivity $N\times N$ matrix C such that $c_{nm}>0$, and P is the $d\times d$ matrix that defines the coupled variables. The complete synchronous state of system (1) is the linear invariant manifold, *synchronization manifold* $D=\{x_1=x_2=\cdots=x_n\}$. The connection of node n with node m is represented by the matrix C element in nth row and mth column.

To demonstrate the ability of our experimental method, let us consider two types of coupling schemes, in-link and out-link, where all links entering or leaving a node have the same weight. (i) The connectivity matrix C_{inf} for the former type, referred to as *informational scheme*, has zero column sums of entries c_{nm}, and the weight w_i of link i is proportional to the total number of links k_m ($m=2,3,\ldots,N$) (N being the total number of the nodes in the network) entering the node m so that the weight sum of all links going into node m is $W_m=\sum_{i=1}^{k_m} w_i=1$, where $w_i=k_m^{-1}$. This coupling scheme can simulate as well, neural, computer, and social networks, where each node receives a certain amount of information regardless of whether it comes from a single or several sources. (ii) The connectivity matrix C_{com} for the latter type, to be called *commodity scheme*, has zero row-sums of the entries c_{nm}. Here, the weight w_i of link i is proportional to the total number of output links l_n ($n=1,\ldots,N$) of node n so that the weight sum of all links leaving the node n is $W_n^*=\sum_{i=1}^{l_n} w_i=1$, i.e., $w_i=l_n^{-1}$. Such a scheme can describe commodities exchanges between factories or transportation companies (airports, bus and train stations, etc.). Each unit generates products to be distributed evenly among other units depending on the number of output connections.

For the simplest case, when the oscillators are coupled by only one variable ($P=1$), say y, the last term in Eq. (1) can be replaced by $\varepsilon(k_m^{-1}\sum_{n=1}^{k_m} y_n - y_m)$ and $\varepsilon(\sum_{n=1}^{k_m} l_n^{-1} y_n - y_m)$ for the informational and commodity schemes, respectively. The networks under consideration are not node balanced, in the sense that the input and output weight sums are not equal. Nevertheless, these networks are very common in practice and their correct operation depends on how well they are synchronized. We are now interested in answering the following questions: what are the differences between synchronization properties of the two network schemes? In particular, what should the optimal coupling be for each network type to obtain the best synchronization between any chosen pair of nodes? Does the optimal coupling depend on network configuration? In order to give some answers, we experimentally study synchronization between every pair of nodes and its dependence on the coupling strength in both network types.

To do so, we use a piecewise linear Rössler-type electronic circuit, the analog version of the following model [21,22]:

$$\frac{dx_n}{d\tau} = -\alpha x_n - z_n - \beta y_n,$$

$$\frac{dy_n}{d\tau} = x_n + \gamma y_n,$$

$$\frac{dz_n}{d\tau} = g(x_n) - z_n, \quad (2)$$

where x_n, y_n, and z_n are the oscillator state variables in node n ($n=1,2,\ldots,N$), $g(x_n)=,\{{}^{0,\ \text{if } x_n\leq 3}_{\mu(x_n-3)\ \text{if } x_n>3}\}$ is the piecewise linear function, and $\tau=t\times 10^4$ s (t being the time). The oscillator generates chaotic wave forms when $\alpha=0.05$, $\beta=0.5$, $\gamma=0.266$, and $\mu=15$. The output variable y_n of node n is coupled with y_m of node m ($m=2,3,\ldots,N, m\neq n$) according to a networks' configuration scheme and the corresponding connectivity matrix. For better illustration, we consider first, a relatively small network formed by only six identical unidirectionally coupled chaotic oscillators. Figure 2 shows the network configuration indicating links' weights for the informational [Fig. 2(a)] and commodity connections [Fig. 2(b)]. The corresponding connectivity matrices are

$$C_{\text{inf}} = \begin{pmatrix} 0 & 1 & 0 & 1/3 & 0 & 1/3 \\ 0 & -1 & 1 & 1/3 & 1/2 & 0 \\ 0 & 0 & -1 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad (3)$$

and

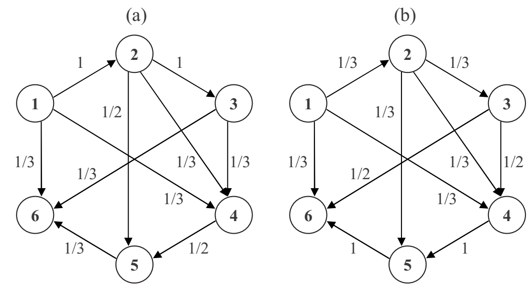

FIG. 2. (a) Informational and (b) commodity coupling schemes for a six-node network.

$$C_{\mathrm{com}} = \begin{pmatrix} -1 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & -1 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & -1 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (3)$$

Note that the diagonal elements $c_{11}=0$ in C_{inf} and $c_{66}=0$ in C_{com} reflect unidirectionality in coupling, i.e., no entrances into node 1 and no outputs from node 6.

Synchronization of complex networks can be quantitatively described by the average synchronization error $\langle e\rangle$ between any pair of oscillators in the network, the cross correlation (K) of their oscillations, and the power spectral density $S(f_0/2)$ at the subharmonic of the fundamental frequency f_0. $\langle e\rangle$ is an important measure in complete synchronization and can be calculated directly from time series, while K and S portray partial synchronization, such as phase or antiphase, and period-doubling synchronization, respectively. Figure 3 presents $\langle e\rangle=\langle y_n-y_m\rangle$ between nodes 1 and $m=2,\ldots,6$ for the networks of 6 as a function of ε for the informational [Figs. 3(a)] and commodity [Fig. 3(b)] schemes. These dependences are not monotonous and their behaviors are determined by a particular type of synchronization. The maxima in $\langle e\rangle$ appear either due to antiphase synchronization or intermittent antiphase synchronization, whereas the minima correspond to phase synchronization. To our knowledge, no experimental evidence for network phase synchronization has been reported although it has already been predicted theoretically [23]. Numerical simulations of Eq. (2) yield good agreement with the experimental results. The advantages of the experimental approach over numerical simulations are that typically it is much faster to generate long chaotic wave forms with an analog system than to obtain them numerically from complex nonlinear models and that the experiments can be performed with a system which model and parameters are not exactly known (e.g., lasers and neurons). The interactions between the network nodes in the experiment are caused by real physical processes which lead to synchronization; the computer only memorizes the output wave forms and models the network configuration.

For larger networks, the calculation time difference between the experimental and numerical methods goes as the numbers of nodes and links, making our approach more practical. The experiments with large networks reveal the main difference between informational and commodity schemes: for the informational type, the optimal coupling (when $\langle e\rangle$ is a minimum) for any pair of oscillators always occurs at $\varepsilon_{opt}=1$ and is independent of the network configuration, while for the commodity type, ε_{opt} is different for each pair of nodes and does indeed depend on the configuration. Note that our setup is time independent; for example, if node 1 is connected with nodes 2 and 1000, the same wave form is used for driving both nodes, and both after transient time (dependent on the initial conditions for each oscillator) may be simultaneously synchronized with node 1.

To visualize the synchronization state of a whole complex network, we construct a very helpful tool, the *network synchronization bifurcation diagram* (NSBD). A particular color (red, green, and blue) is assigned to each property and each line of pixels represents the corresponding oscillator synchronization state with respect to the reference one. We choose red for $\langle e\rangle$, green for K, and blue for S, and we take ε as a control parameter to obtain the NSBD shown in Fig. 4. The gray value for each color is in the range [0, 255] indicating the corresponding minimum and maximum values. Complete synchronization yields the green color because $\langle e\rangle=0$, $K=1$, and $S=0$, an asynchronous state results in red because $K\approx 0$, $S=0$ and $\langle e\rangle$ takes a big value, while period-doubling synchronization [22] manifests itself with the blue color. Figure 4 shows NSBDs for two 1000-oscillator random networks, having the same adjacency matrix but coupled either with informational [Fig. 4(a)] or commodity schemes [Fig. 4(b)]. Similar NSBDs can be constructed for any other node $n>1$, in this case the number of lines in the image will be equal to $N-n$. The spatial-pattern formation in NSBD gives significant information on the evolution of the

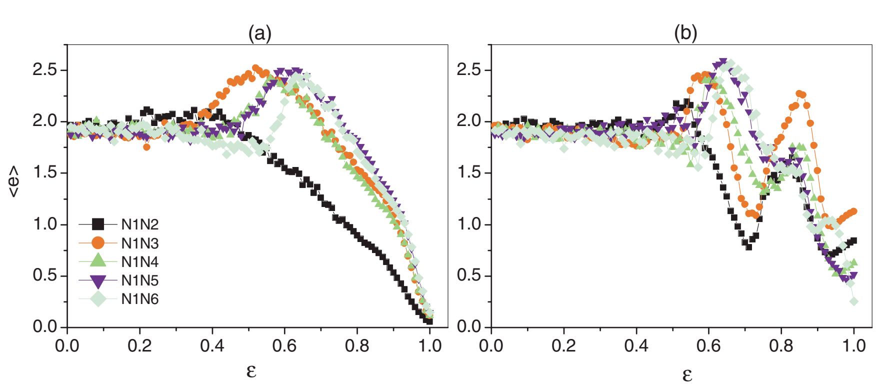

FIG. 3. (Color online) Average synchronization error between node 1 and other nodes for networks of six Rössler oscillators coupled with (a) informational and (b) commodity schemes, as a function of coupling strength. The node m with which the synchronization error is measured is indicated in the left down corner.

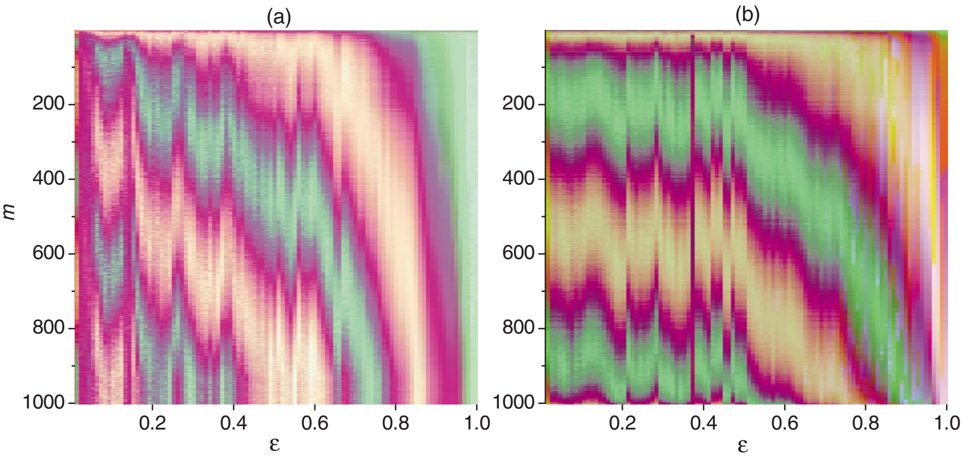

FIG. 4. (Color online) Synchronization bifurcation diagrams of a 1000-oscillator random network with maximum 100 links per node, coupled with (a) informational and (b) commodity schemes. The pixels' colors display synchronization states of 999 oscillators $m=2,\ldots,1000$ with respect to oscillator $n=1$ for 100 coupling strengths $\varepsilon=0.01,\ldots,1$. The red (gray), green (light gray), and blue (dark gray) colors show, respectively, the contributions of $\langle e\rangle$, K, and S.

whole network as the coupling changes: yellow regions indicate anti-synchronization, while blue regions tell us of a period-doubling regime; an abrupt transformation of the color means a sudden change in synchronization type and may well be a bifurcation. Implicitly is the previous work of making a *synchronization image* that did show the state of synchronization of the whole network at a fixed coupling parameter. In this $N\times N$-pixel image, each pixel (n,m) color represents the synchronization state between nodes n and m; one has to be aware that the green diagonal has no meaning.

In conclusion, we have proposed an experimental method to study unidirectional network synchronization using a single oscillator that is a complete analog real physical system (it could also have been biological, chemical, or medical). The Rössler oscillator was chosen only for a better demonstration and a clearer idea of how this experimental arrangement works. Although our approach does not allow the study of synchronization in real time, both the generation and the interaction of the oscillator wave forms happen outside the computer, and therefore all physical processes are completely analog, so that no any previous knowledge about the theoretical model is required. This is the great strength of the method, the oscillator could actually be a black box which generates any wave form. This setup enabled us to highlight the difference between an informational and a commodity coupling schemes. The visualization of network synchronization introduced in this work is not only illustrative but is in fact an effective tool for analysis, design, and optimization of complex networks. Nowadays, the experimental approach proposed opens a clear possibility for studying synchronization of large complex networks of a desired configuration. The comparison of our approach with multiples delayed feedback methods might be of interest for future investigation.

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